

Frugal Bribery in Voting

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Abstract

Bribery in elections is an important problem in computational social choice theory. We introduce and study two important special cases of the classical \$BRIBERY problem, namely, FRUGAL-BRIBERY and FRUGAL-\$BRIBERY where the briber is frugal in nature. By this, we mean that the briber is only able to influence voters who benefit from the suggestion of the briber. More formally, a voter is *vulnerable* if the outcome of the election improves according to her own preference when she accepts the suggestion of the briber. In the FRUGAL-BRIBERY problem, the goal is to make a certain candidate win the election by changing *only* the vulnerable votes. In the FRUGAL-\$BRIBERY problem, the vulnerable votes have prices and the goal is to make a certain candidate win the election by changing only the vulnerable votes, subject to a budget constraint. We further formulate two natural variants of the FRUGAL-\$BRIBERY problem namely UNIFORM-FRUGAL-\$BRIBERY and NONUNIFORM-FRUGAL-\$BRIBERY where the prices of the vulnerable votes are, respectively, all the same or different. The FRUGAL-BRIBERY problem turns out to be a special case of sophisticated \$BRIBERY as well as SWAP-BRIBERY problems. Whereas the FRUGAL-\$BRIBERY problem turns out to be a special case of the \$BRIBERY problem.

We show that the FRUGAL-BRIBERY problem is polynomial time solvable for the k-approval, k-veto, and plurality with run off voting rules for unweighted elections. These results establish success in finding practically appealing as well as polynomial time solvable special cases of the sophisticated \$BRIBERY and SWAP-BRIBERY problems. On the other hand, we show that the FRUGAL-BRIBERY problem is NP-complete for the Borda voting rule and the FRUGAL-\$BRIBERY problem is NP-complete for most of the voting rules studied here barring the plurality and the veto voting rules for unweighted elections. Our hardness results of the FRUGAL-BRIBERY and the FRUGAL-\$BRIBERY problems thus subsumes and strengthens the hardness results of the \$BRIBERY problem from the literature. For the weighted elections, we show that the FRUGAL-BRIBERY problem is NP-complete for all the voting rules studied here except the plurality voting rule even when the number of candidates is as low as 3 (for the STV and the plurality with run off voting rules) or 4 (for the maximin, the Copeland ^{α} with $\alpha \in [0, 1]$, and the simplified Bucklin voting rules). In our view, the fact that the simplest FRUGAL-BRIBERY problem becomes computationally intractable for many important voting rules (except the plurality voting rule) even with

very few candidates is surprising as well as interesting.

Keywords: Computational social choice; voting; bribery; frugal; manipulation; algorithm; theory.

1. Introduction

In a typical voting scenario, we have a set of candidates and a set of voters reporting their *preferences or votes* which are complete rankings over the candidates. A *voting rule* is a procedure that, given a collection of votes, chooses one candidate as the winner. A set of votes over a set of candidates along with a voting rule is called an election.

Activities that try to influence voter opinions, in favor of specific candidates, are very common during the time that an election is in progress. For example, in a political election, candidates often conduct elaborate campaigns to promote themselves among a general or targeted audience. Similarly, it is not uncommon for people to protest against, or rally for, a national committee or court that is in the process of approving a particular policy. An extreme illustration of this phenomenon is *bribery* — here, the candidates may create financial incentives to sway the voters. Of course, the process of influencing voters may involve costs even without the bribery aspect; for instance, a typical political campaign or rally entails considerable expenditure.

All situations involving a systematic attempt to influence voters usually have the following aspects: an external agent, a candidate that the agent would like to be the winner, a budget constraint, a cost model for a change of vote, and knowledge of the existing election. The formal computational problem that arises from these inputs is the following: is it possible to make a distinguished candidate win the election in question by incurring a cost that is within the budget? This question, with origins in Faliszewski et al. [FHH06, FHH09, FHHR09], has been subsequently studied intensely in computational social choice literature. In particular, bribery has been studied under various cost models, for example, uniform price per vote which is known as \$BRIBERY [FHH06], nonuniform price per vote [Fal08], nonuniform price per shift of the distinguished candidate per vote which is called SHIFT BRIBERY, nonuniform price per swap of candidates per vote which is called SWAP BRIBERY [EFS09a]. A closely related problem known as campaigning has been studied for various vote models, for example, truncated ballots [BFLR12], soft constraints [PRV13], CP-nets [DK15], combinatorial domains [MPVR12] and probabilistic lobbying [BEF⁺14]. The bribery problem has also been studied under voting rule uncertainty [EHH14]. Faliszewski et al. [FRRS14] study the complexity of bribery in simplified Bucklin and Fallback voting rules. Xia [Xia12] studies destructive bribery, where the goal of the briber is to change the winner by changing minimum number of votes. Dorn et al. [DS12] studies the parameterized complexity of the SWAP BRIBERY problem and Brederick et al. [BCF⁺14] explores the parameterized complexity of the SHIFT BRIBERY problem for a wide range of parameters. We recall again that the costs and the budgets involved in all the bribery problems above need not necessarily correspond to actual money

traded between voters and candidates. They may correspond to any cost in general, for example, the amount of effort or time that the briber needs to spend for each voter.

1.1. Motivation

In this work, we propose an effective cost model for the bribery problem. Even the most general cost models that have been studied in the literature fix absolute costs per voter-candidate combination, with no specific consideration to the voters' opinions about the current winner and the distinguished candidate whom the briber wants to be the winner. In our proposed model, a change of vote is relatively easier to effect if the change causes an outcome that the voter would find desirable. Indeed, if the currently winning candidate is, say, a , and a voter is (truthfully) promised that by changing her vote from $c \succ d \succ a \succ b$ to $d \succ b \succ c \succ a$, the winner of the election would change from a to d , then this is a change that the voter is likely to be happy to make. While the change does not make her most favorite candidate win the election, it does improve the result from her point of view. Thus, given the circumstances (namely that of her least favorite candidate winning the election), the altered vote serves the voter better than the original one.

We believe this perspective of voter influence is an important one to study. The cost of a change of vote is proportional to the nature of the outcome that the change promises — the cost is low or nil if the change results in a better outcome with respect to the voter's original ranking, and high or infinity otherwise. A frugal agent only approaches voters of the former category, thus being able to effectively bribe with minimal or no cost. Indeed the behavior of agents in real life is often frugal. For example, consider campaigners in favor of a relatively smaller party in a political election. They may actually target only vulnerable voters due to lack of human and other resources they have at their disposal.

More formally, let c be the winner of an election and p (other than c) the candidate whom the briber wishes to make the winner of the election. Now the voters who prefer c to p will be reluctant to change their votes, and we call these votes *non-vulnerable with respect to p* — we do not allow these votes to be changed by the briber, which justifies the *frugal* nature of the briber. On the other hand, if a voter prefers p to c , then it may be very easy to convince her to change her vote if doing so makes p win the election. We name these votes *vulnerable with respect to p* . When the candidate p is clear from the context, we simply call these votes non-vulnerable and vulnerable, respectively.

The computational problem is to determine whether there is a way to make a candidate p win the election by changing *only* those votes that are vulnerable with respect to p . We call this problem FRUGAL-BRIBERY. Note that there is no cost involved in the FRUGAL-BRIBERY problem — the briber does not incur any cost to change the votes of the vulnerable votes. We also extend this basic model to a more general setting where each vulnerable vote has a certain nonnegative integer price which may correspond to the effort involved in approaching these voters and convincing them to change their votes. We also allow for the specification of a budget constraint, which can be used to enforce auxiliary constraints. This leads us to define the FRUGAL-\$BRIBERY problem, where we are required to find a subset of vulnerable votes with a total cost that is within a given budget, such that these votes can be changed in some way to make the candidate p win the election. Note that the

FRUGAL-\$BRIBERY problem can be either uniform or nonuniform depending on whether the prices of the vulnerable votes are all identical or different. If not mentioned otherwise, the prices of the vulnerable votes will be assumed to be nonuniform. We remind that the briber is not allowed to change the non-vulnerable votes in both the FRUGAL-BRIBERY and the FRUGAL-\$BRIBERY problems.

1.2. Contributions

Our primary contribution in this paper is to formulate and study two important and natural models of bribery which turn out to be special cases of the well studied \$BRIBERY problem in elections. Indeed, the FRUGAL-\$BRIBERY problem and, more importantly, the FRUGAL-BRIBERY problem are very restricted yet practically appealing cases of the \$BRIBERY problem.

Our Results for Unweighted Elections

We have the following polynomial time algorithms for unweighted elections. These results show that the FRUGAL-BRIBERY problem is computationally tractable for some voting rules for which the \$BRIBERY problem is NP-complete as observed for the k-approval with $k \geq 3$ [Lin11], simplified Bucklin [FRRS15]. We summarize the results in Table 1.

- The FRUGAL-BRIBERY problem is in P for the k-approval, simplified Bucklin, and plurality with runoff voting rules. Also, the FRUGAL-\$BRIBERY problem is in P for the plurality and veto voting rules.
- The FRUGAL-\$BRIBERY problem is in P for the k-approval, simplified Bucklin, and plurality with runoff voting rules when the budget is a constant [Theorem 4].

We have the following intractability results for the FRUGAL-BRIBERY problem and the FRUGAL-\$BRIBERY problem for unweighted elections. Our hardness results of the FRUGAL-BRIBERY and the FRUGAL-\$BRIBERY problems below thus subsume and strengthen the hardness results of the \$BRIBERY problem from the literature.

- The FRUGAL-BRIBERY problem is NP-complete for the Borda voting rule [Theorem 1]. The FRUGAL-\$BRIBERY is NP-complete for the k-approval for any constant $k \geq 5$ [Theorem 2], k-veto for any constant $k \geq 3$ [Theorem 3], and a wide class of scoring rules [Theorem 5] even if the price of every vulnerable vote is either 1 or ∞ . Moreover, the UNIFORM-FRUGAL-\$BRIBERY is NP-complete for the Borda voting rule even if all the vulnerable votes have a uniform price of 1 and the budget is 2 [Theorem 6].
- The FRUGAL-\$BRIBERY problem is NP-complete for the Borda, maximin, Copeland, and STV voting rules [Observation 3].

Our Results for Weighted Elections

We have the following results for weighted elections. We observe that, barring a few exceptions, even the most restrictive FRUGAL-BRIBERY problem is NP-complete even when we have only 3 or 4 candidates.

- The FRUGAL-BRIBERY problem is in P for the maximin and Copeland voting rules when we have only 3 candidates [Observation 4], and for the plurality voting rule for any number of candidates [Theorem 7].
- The FRUGAL-BRIBERY problem is NP-complete for the STV [Theorem 10], plurality with runoff [Corollary 1], and every scoring rule except the plurality voting rule [Observation 5] for 3 candidates. The FRUGAL-\$BRIBERY problem is NP-complete for the plurality voting rule for 3 candidates [Theorem 8].
- When we have only 4 candidates, the FRUGAL-BRIBERY problem is NP-complete for the maximin [Theorem 9], simplified Bucklin [Theorem 12], and Copeland [Theorem 11] rules.

1.3. Related Work

The pioneering work of Faliszewski et al. [FHH06] defined and studied the \$BRIBERY problem wherein, the input is a set of votes with prices for each vote and the goal is to make some distinguished candidate win the election, subject to a budget constraint of the briber. The FRUGAL-\$BRIBERY problem is the \$BRIBERY problem with the restriction that the price of every non-vulnerable vote is infinite. Also, the FRUGAL-BRIBERY problem is a special case of the FRUGAL-\$BRIBERY problem. Hence, whenever the \$BRIBERY problem is computationally easy in a setting, both the FRUGAL-BRIBERY and the FRUGAL-\$BRIBERY problems are also computationally easy (see Proposition 1 for a more formal proof). However, the \$BRIBERY problem is computationally intractable in most of the settings. This makes the study of important special cases such as FRUGAL-BRIBERY and FRUGAL-\$BRIBERY, interesting. Elkind et al. [EFS09b] introduced and studied the SWAP-BRIBERY problem where we have a more sophisticated cost model specifying cost of swapping every pair of candidates for every vote. It turns out that the FRUGAL-BRIBERY problem is a special case of the SWAP-BRIBERY problem (see Proposition 3 for a formal proof). We note that a notion similar to vulnerable votes has been studied in the context of dominating manipulation by Conitzer et al. [CWX11]. Hazon et al. [HLK13] introduced and studied PERSUASION and k-PERSUASION problems for the plurality, veto, k-approval, Bucklin, and Borda voting rules in unweighted elections only. In the PERSUASION and k-PERSUASION problems an external agent suggests votes to vulnerable voters which are beneficial for the vulnerable voters as well as the external agent. It turns out that the PERSUASION and the k-PERSUASION problems Turing reduce to the FRUGAL-BRIBERY and the FRUGAL-\$BRIBERY problems respectively (see Proposition 4). Therefore, the polynomial time algorithms we propose in this work imply polynomial time algorithms for the persuasion analog. On the other hand,

Voting Rules	Unweighted		Weighted	
	FRUGAL-BRIBERY	FRUGAL-\$BRIBERY	FRUGAL-BRIBERY	FRUGAL-\$BRIBERY
Plurality	P [Observation 1]	P [Observation 2]	P [Theorem 7]	NP-complete [Theorem 8]
Veto	P [Observation 1]	P [Observation 2]	NP-complete [Observation 5]	NP-complete [Observation 5]
k-approval	P [Observation 1]	NP-complete [*] [Theorem 2]	NP-complete [◇] [Observation 5]	NP-complete [Observation 5]
k-veto	P [Observation 1]	NP-complete [*] [Theorem 3]	NP-complete [◇] [Observation 5]	NP-complete [Observation 5]
Borda	NP-complete [Theorem 1]	NP-complete [†] [Theorem 5]	NP-complete [Observation 5]	NP-complete [Observation 5]
Runoff	P [Observation 1]	?	NP-complete [Corollary 1]	NP-complete [Corollary 1]
Maximin	?	NP-complete [Observation 3]	NP-complete [Theorem 9]	NP-complete [Theorem 9]
Copeland	?	NP-complete [Observation 3]	NP-complete [Theorem 11]	NP-complete [Theorem 11]
STV	?	NP-complete [Observation 3]	NP-complete [Theorem 10]	NP-complete [Theorem 10]

Table 1: ^{*}- The result holds for $k \geq 3$. [†]- The result holds for a much wider class of scoring rules. [◇]- The results do not hold for the plurality voting rule. ?- The problem is open.

since the reduction in Proposition 4 from PERSUASION to FRUGAL-BRIBERY is a Turing reduction, the existing NP-completeness results for the persuasion problems do not imply NP-completeness results for the corresponding frugal bribery variants. We refer to the book by Rogers [RR67] for Turing reductions.

Organization. The rest of the paper is organized as follows. We first establish the setup and general notions in Section 2. Next we present our results for unweighted elections in Section 3 and our results for the weighted elections Section 4. Finally we conclude in Section 5. A Preliminary version of this work appeared at AAAI-16 [DMN16a].

2. Preliminaries

Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be the set of all *voters* and $\mathcal{C} = \{c_1, \dots, c_m\}$ the set of all *candidates*. Each voter v_i 's *vote* is a *preference* \succ_i over the candidates which is a linear order over \mathcal{C} . For example, for two candidates a and b , $a \succ_i b$ means that the voter v_i prefers a to b . Let \succ be a vote over \mathcal{C} , $x \in \mathcal{C}$ be a candidate, and k be a positive integer. We say that x is placed at the k^{th} position in the vote \succ if there are exactly $k - 1$ candidates in $\mathcal{C} \setminus \{x\}$ who are preferred over x in \succ ; that is, $|\{y \in \mathcal{C} \setminus \{x\} : y \succ x\}| = k - 1$. We say that x is placed at the top or the first position of \succ if x is preferred over every other candidate $y \in \mathcal{C} \setminus \{x\}$. We say that x is placed at the bottom or the last position of \succ if every candidate $y \in \mathcal{C} \setminus \{x\}$ other than x is preferred over x in \succ . We say that x is placed at the k^{th} position from the bottom or the last position if x is preferred over exactly $k - 1$ candidates $y \in \mathcal{C} \setminus \{x\}$ other than x ; that is, $|\{y \in \mathcal{C} \setminus \{x\} : x \succ y\}| = k - 1$. In this paper, whenever we do not specify the order among a set of candidates while describing a vote, the statement/proof is correct in whichever way we fix the order among them. We denote the set $\{0, 1, 2, \dots\}$ by \mathbb{N} , $\mathbb{N} \setminus \{0\}$ by \mathbb{N}^+ , and $\{1, \dots, k\}$ by $[k]$, for any positive integer k . We denote the set of all linear orders over \mathcal{C} by $\mathcal{L}(\mathcal{C})$. Hence, $\mathcal{L}(\mathcal{C})^n$ denotes the set of all n -voters' preference profiles $\succ_{[n]} = (\succ_1, \dots, \succ_n)$. Let \uplus denote the disjoint union of sets. A map $r_c : \uplus_{n, |\mathcal{C}| \in \mathbb{N}^+} \mathcal{L}(\mathcal{C})^n \rightarrow 2^{\mathcal{C}} \setminus \{\emptyset\}$ is called a *voting correspondence*. A map $t : 2^{\mathcal{C}} \setminus \{\emptyset\} \rightarrow \mathcal{C}$ is called a *tie breaking rule*. A commonly used tie breaking rule is the *lexicographic* tie breaking rule where ties are broken according to a predetermined preference $\succ_t \in \mathcal{L}(\mathcal{C})$. A *voting rule* is $r = t \circ r_c$, where \circ denotes the composition of maps.

Remark. We note that, in the literature, the definition of a voting rule is usually defined as what we are referring to as a voting correspondence in the above. In particular, the tie-breaking rule is often left out from the definition. We choose to only deal with voting rules that lead to unique winners by definition, because of our notion of vulnerable votes. However, the notion of vulnerable votes can be generalized in natural ways (say, for instance, that a vote is vulnerable if it prefers the desired candidate over all the current winners; or at least one of them). As long as we require p to be the unique winner of the bribed profile, our proofs will carry over to the more general setting. We use tie-breaking rules mostly for ease of presentation.

In many settings, the voters may have positive integer weights. Such an election is called a weighted election. The winner of a weighted election is defined to be the winner

of the unweighted election where each vote is replaced by as many copies of the vote as its weight. We remark that for all the voting rules studied here, the winner of any weighted election can be computed in polynomial amount of time. We assume the elections to be unweighted, if not stated otherwise. Given an election E , we can construct a directed weighted graph G_E , called the *weighted majority graph*, from E . The set of vertices in G_E is the set of candidates in E . For any two candidates x and y , the weight of the edge (x, y) is $D_E(x, y) = N_E(x, y) - N_E(y, x)$, where $N_E(a, b)$ is the number of voters who prefer candidate a to b . A candidate x is called the *Condorcet winner* in an election E if $D_E(x, y) > 0$ for every other candidate $y \neq x$. A voting rule is called *Condorcet consistent* if it selects the Condorcet winner as the winner of the election whenever it exists. Some examples of common voting correspondences are as follows.

- **Positional scoring rules:** A collection of m -dimensional vectors $\vec{s}_m = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ and $\alpha_1 > \alpha_m$ for every $m \in \mathbb{N}$ naturally defines a voting rule — a candidate gets score α_i from a vote if it is placed at the i^{th} position, and the score of a candidate is the sum of the scores it receives from all the votes. The winners are the candidates with maximum score. Scoring rules remain unchanged if we multiply every α_i by any constant $\lambda > 0$ and/or add any constant μ . Hence, we assume without loss of generality that for any score vector \vec{s}_m , there exists a j such that $\alpha_j - \alpha_{j+1} = 1$ and $\alpha_k = 0$ for all $k > j$. We call such a \vec{s}_m a *normalized score vector*.

If α_i is 1 for $i \in [k]$ and 0 otherwise, then, we get the *k-approval* voting rule. For the *k-veto* voting rule, α_i is 0 for $i \in [m - k]$ and -1 otherwise. 1-approval is called the *plurality* voting rule and 1-veto is called the *veto* voting rule. If $\alpha_i = m - i$ for every $i \in [m]$, then we get the *Borda* voting rule.

- **Maximin:** The maximin score of a candidate x is $\min_{y \neq x} D_E(x, y)$. The winners are the candidates with maximum maximin score.
- **Copeland $^\alpha$:** Given $\alpha \in [0, 1]$, the Copeland $^\alpha$ score of a candidate x is $|\{y \neq x : D_E(x, y) > 0\}| + \alpha |\{y \neq x : D_E(x, y) = 0\}|$. The winners are the candidates with maximum Copeland $^\alpha$ score. If not mentioned otherwise, we will assume α to be zero.
- **Simplified Bucklin:** A candidate x 's simplified Bucklin score is the minimum number ℓ such that more than half of the voters rank x in their top ℓ positions. The winners are the candidates with lowest simplified Bucklin score.
- **Plurality with runoff:** The top two candidates according to the plurality scores are selected first. The pairwise winner of these two candidates is selected as the winner. This rule is often called the runoff voting rule.
- **Single Transferable Vote:** In Single Transferable Vote (STV), a candidate with the least plurality score is dropped from the election and its votes are transferred to the

next preferred candidate. If two or more candidates receive the least plurality score, then a tie breaking rule is used. The candidate that remains after $(m - 1)$ rounds is the winner.

Among the above voting correspondences along with any arbitrary lexicographic tie-breaking rule, only the maximin and the Copeland voting rules are Condorcet consistent.

We use the notation $A \leq_P B$ to denote that the problem A polynomial time many-to-one reduces to the problem B .

2.1. Problem Definition

In all the definitions below, r is a fixed voting rule. We define the notion of vulnerable votes as follows. Intuitively, the vulnerable votes are those votes whose voters can easily be persuaded to change their votes since doing so will result in an outcome that those voters prefer over the current one.

Definition 1. (*Vulnerable votes*)

Given a voting rule r , a set of candidates \mathcal{C} , a profile of votes $\succ = (\succ_1, \dots, \succ_n)$, and a distinguished candidate p , we say a vote \succ_i is p -vulnerable if $p \succ_i r(\succ)$.

Recall that, whenever the distinguished candidate is clear from the context, we drop it from the notation. With the above definition of vulnerable votes, we formally define the FRUGAL-BRIBERY problem as follows. Intuitively, the problem is to determine whether a particular candidate can be made winner by changing only the vulnerable votes.

r -FRUGAL-BRIBERY

Input: A set \mathcal{C} of candidates, a preference profile $\succ = (\succ_1, \dots, \succ_n)$ over \mathcal{C} , and a candidate p .

Question: Is there a way to make p win the election according to the voting rule r by changing only the p -vulnerable votes?

We denote an arbitrary instance of r -FRUGAL-BRIBERY by (\mathcal{C}, \succ, p) . Next we generalize the FRUGAL-BRIBERY problem to the FRUGAL-\$BRIBERY problem which involves prices for the vulnerable votes and a budget for the briber. Intuitively, the price of a vulnerable vote v is the cost the briber incurs to change the vote v .

r-FRUGAL-\$BRIBERY

Input: A set \mathcal{C} of candidates, a preference profile $\succ = (\succ_1, \dots, \succ_n)$ over \mathcal{C} , a candidate p , a finite budget $b \in \mathbb{N}$, and a price function $c : [n] \rightarrow \mathbb{N} \cup \{\infty\}$ such that $c(i) = \infty$ if \succ_i is not a p -vulnerable vote.

Question: Do there exist p -vulnerable votes $\succ_{i_1}, \dots, \succ_{i_\ell} \in \succ$ and votes $\succ'_{i_1}, \dots, \succ'_{i_\ell} \in \mathcal{L}(\mathcal{C})$ such that:

- (a) the total cost of the chosen votes is within the budget, that is, $\sum_{j=1}^{\ell} c(i_j) \leq b$, and
- (b) the new votes make the desired candidate win according to the voting rule r , that is, $r(\succ_{[n] \setminus \{i_1, \dots, i_\ell\}}, \succ'_{i_1}, \dots, \succ'_{i_\ell}) = p$.

The special case of the problem when the prices of all the vulnerable votes are the same is called UNIFORM-FRUGAL-\$BRIBERY. We refer to the general version as NONUNIFORM-FRUGAL-\$BRIBERY. If not specified, FRUGAL-\$BRIBERY refers to the nonuniform version. We denote an arbitrary instance of r -FRUGAL-\$BRIBERY by $(\mathcal{C}, \succ, p, c(\cdot))$. The above problems are important special cases of the well studied \$BRIBERY problem. Also, the COALITIONAL-MANIPULATION problem [BTT89, CSL07], one of the classic problems in computational social choice theory, turns out to be a special case of the FRUGAL-\$BRIBERY problem [see Proposition 1]. For the sake of completeness, we include the definitions of these problems here.

r-\$BRIBERY [FHH09]

Input: A set \mathcal{C} of candidates, a preference profile $\succ = (\succ_1, \dots, \succ_n)$ over \mathcal{C} , a candidate p , a price function $c : [n] \rightarrow \mathbb{N} \cup \{\infty\}$, and a budget $b \in \mathbb{N}$.

Question: Do there exist votes $\succ_{i_1}, \dots, \succ_{i_\ell} \in \succ$ and votes $\succ'_{i_1}, \dots, \succ'_{i_\ell} \in \mathcal{L}(\mathcal{C})$ such that:

- (a) the total cost of the chosen votes is within the budget, that is, $\sum_{j=1}^{\ell} c(i_j) \leq b$, and
- (b) the new votes make the desired candidate win according to the voting rule r , that is, $r(\succ_{[n] \setminus \{i_1, \dots, i_\ell\}}, \succ'_{i_1}, \dots, \succ'_{i_\ell}) = p$.

r-COALITIONAL-MANIPULATION [BTT89, CSL07]

Input: A set \mathcal{C} of candidates, a preference profile $\succ^t = (\succ_1, \dots, \succ_n)$ of truthful voters over \mathcal{C} , an integer ℓ encoded in unary, and a distinguished candidate p .

Question: Does there exist an ℓ voter preference profile \succ^ℓ such that the candidate p wins uniquely (does not tie with any other candidate) in the profile (\succ^t, \succ^ℓ) according to the voting rule r ?

The following proposition shows relations among the above problems. Proposition 1, 2 and 4 below hold for both weighted and unweighted elections.

Proposition 1. *For every voting rule, $\text{FRUGAL-BRIBERY} \leq_P \text{UNIFORM-FRUGAL-}\$ \text{BRIBERY} \leq_P \text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY} \leq_P \$ \text{BRIBERY}$. Also, $\text{COALITIONAL-MANIPULATION} \leq_P \text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY}$.*

Proof. In the reductions below, let us assume that the election to start with is a weighted election. Since we do not change the weights of any vote in the reduction and since there is a natural one to one correspondence between the votes of the original instance and the reduced instance, the proof also works for unweighted elections.

Given a FRUGAL-BRIBERY instance, we construct a $\text{UNIFORM-FRUGAL-}\$ \text{BRIBERY}$ instance by defining the price of every vulnerable vote to be zero and the budget to be zero. Clearly, the two instances are equivalent. Hence, $\text{FRUGAL-BRIBERY} \leq_P \text{UNIFORM-FRUGAL-}\$ \text{BRIBERY}$.

$\text{UNIFORM-FRUGAL-}\$ \text{BRIBERY} \leq_P \text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY} \leq_P \$ \text{BRIBERY}$ follows from the fact that $\text{UNIFORM-FRUGAL-}\$ \text{BRIBERY}$ is a special case of $\text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY}$ which in turn is a special case of $\$ \text{BRIBERY}$.

Given a $\text{COALITIONAL-MANIPULATION}$ instance, we construct a $\text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY}$ instance as follows. Let p be the distinguished candidate of the manipulators and $\succ_f = p \succ \text{others}$ be any arbitrary but fixed ordering of the candidates given in the $\text{COALITIONAL-MANIPULATION}$ instance. Without loss of generality, we can assume that p does not win if all the manipulators vote \succ_f (Since this is a polynomially checkable case of $\text{COALITIONAL-MANIPULATION}$). We define the vote of the manipulators to be \succ_f , the distinguished candidate of the campaigner to be p , the budget of the campaigner to be zero, the price of the manipulators to be zero (notice that all the manipulators' votes are p -vulnerable), and the price of the rest of the vulnerable votes to be one. Clearly, the two instances are equivalent. Hence, $\text{COALITIONAL-MANIPULATION} \leq_P \text{NONUNIFORM-FRUGAL-}\$ \text{BRIBERY}$. \square

Also, the FRUGAL-BRIBERY problem reduces to the $\text{COALITIONAL-MANIPULATION}$ problem by simply making all vulnerable votes to be manipulators.

Proposition 2. *For every voting rule, $\text{FRUGAL-BRIBERY} \leq_P \text{COALITIONAL-MANIPULATION}$.*

The FRUGAL-BRIBERY problem also reduces to the SWAP-BRIBERY problem as proved below.

Proposition 3. *For every voting rule, $\text{FRUGAL-BRIBERY} \leq_P \text{SWAP-BRIBERY}$.*

Proof. Given an arbitrary instance of the FRUGAL-BRIBERY problem, we define the SWAP-BRIBERY instance simply by defining the cost of every swap in vulnerable votes to be 0, the cost of every swap in non-vulnerable votes to be 1, and the budget to be 0. \square

We can also establish the following relation between the PERSUASION (respectively $k\text{-PERSUASION}$) problem and the FRUGAL-BRIBERY (respectively $\text{FRUGAL-}\$ \text{BRIBERY}$) problem. The persuasions differ from the corresponding frugal bribery variants in that the briber has her own preference order, and desires to improve the outcome of the election with respect to her preference order. The following proposition is immediate from the definitions of the problems.

Proposition 4. *For every voting rule, there is a Turing reduction from PERSUASION (respectively k -PERSUASION) to FRUGAL-BRIBERY (respectively FRUGAL-\$BRIBERY).*

Proof. Given an algorithm for the FRUGAL-BRIBERY problem, we iterate over all possible distinguished candidates to have an algorithm for the persuasion problem.

Given an algorithm for the FRUGAL-\$BRIBERY problem, we iterate over all possible distinguished candidates and fix the price of the corresponding vulnerables to be one to have an algorithm for the k -persuasion problem. \square

3. Results for Unweighted Elections

Now we present the results for unweighted elections. We begin with some easy observations that follow from known results.

Observation 1. *The FRUGAL-BRIBERY problem is in P for the k -approval voting rule for any k , simplified Bucklin, and plurality with runoff voting rules.*

Proof. The COALITIONAL-MANIPULATION problem is in P for these voting rules [XZP⁺09]. Hence, the result follows from Proposition 2. \square

Observation 2. *The FRUGAL-\$BRIBERY problem is in P for the plurality and veto voting rules.*

Proof. The \$BRIBERY problem is in P for the plurality [FHH06] and veto [Fal08] voting rules. Hence, the result follows from Proposition 1. \square

Observation 3. *The FRUGAL-\$BRIBERY problem is NP-complete for Borda, maximin, Copeland, and STV voting rules.*

Proof. The COALITIONAL-MANIPULATION problem is NP-complete for the above voting rules. Hence, the result follows from Proposition 1. \square

We now present our main results. We begin with showing that the FRUGAL-BRIBERY problem for the Borda voting rule. To this end, we reduce from the PERMUTATION SUM problem, which is known to be NP-complete [YHL04]. The PERMUTATION SUM problem is defined as follows.

PERMUTATION SUM

Input: n integers $X_i, i \in [n]$ with $1 \leq X_i \leq 2n$ for every $i \in [n]$ and $\sum_{i=1}^n X_i = n(n+1)$.

Question: Do there exist two permutations π and σ of $[n]$ such that $\pi(i) + \sigma(i) = X_i$ for every $i \in [n]$?

We now prove that the FRUGAL-BRIBERY problem is NP-complete for the Borda voting rule, by a reduction from PERMUTATION SUM. Our reduction is inspired by the reduction used by Davies et al. [DKNW11] and Betzler et al. [BNW11] to prove NP-completeness of the COALITIONAL-MANIPULATION problem for the Borda voting rule.

Theorem 1. *The FRUGAL-BRIBERY problem is NP-complete for the Borda voting rule.*

Proof. The problem is clearly in NP. To show NP-hardness, we reduce an arbitrary instance of the PERMUTATION SUM problem to the FRUGAL-BRIBERY problem for the Borda voting rule. Let (X_1, \dots, X_n) be an instance of the PERMUTATION SUM problem. Without loss of generality, let us assume that n is an odd integer – if n is an even integer, then we consider the instance $(X_1, \dots, X_n, X_{n+1} = 2(n+1))$ which is clearly equivalent to the instance (X_1, \dots, X_n) .

We define a FRUGAL-BRIBERY instance $(\mathcal{C}, \mathcal{P}, p)$ as follows. The candidate set is:

$$\mathcal{C} = \mathcal{X} \uplus D \uplus \{p, c\}, \text{ where } \mathcal{X} = \{x_i : i \in [n]\} \text{ and } |D| = 3n - 1$$

Note that the total number of candidates is $4n + 1$, and therefore the Borda score of a candidate placed at the top position is $4n$.

Before describing the votes, we give an informal overview of how the reduction will proceed. The election that we define will consist of exactly two vulnerable votes. Note that when placed at the top position in these two votes, the distinguished candidate p gets a score of $8n$ ($4n$ from each vulnerable vote). We will then add non-vulnerable votes, which will be designed to ensure that, among them, the score of x_i is $8n - X_i$ more than the score of the candidate p . Using the “dummy candidates”, we will also be able to ensure that the candidates x_i receive (without loss of generality) scores between 1 and n from the modified vulnerable votes.

Now suppose these two vulnerable votes can be modified to make p win the election. Let s_1 and s_2 be the scores that x_i obtains from these altered vulnerable votes. It is clear that for p to emerge as a winner, $s_1 + s_2$ must be at most X_i . Since the Borda scores for the candidates in \mathcal{X} range from 1 to n in the altered vulnerable votes, the total Borda score that all the candidates in \mathcal{X} can accumulate from two altered vulnerable votes is $n(n+1)$. On the other hand, since the sum of the X_i ’s is also $n(n+1)$, it turns out that $s_1 + s_2$ must in fact be equal to X_i for the candidate p to win. From this point, it is straightforward to see how the permutations σ and π can be inferred from the modified vulnerable votes: $\sigma(i)$ is given by the score of the candidate x_i from the first vote, while $\pi(i)$ is the score of the candidate x_i from the second vote. These functions turn out to be permutations because these n candidates receive n distinct scores from these votes.

We are now ready to describe the construction formally. We remark that instead of $8n - X_i$, as described above, we will maintain a score difference of either $8n - X_i$ or $8n - X_i - 1$ depending on whether X_i is even or odd respectively — this is a minor technicality that comes from the manner in which the votes are constructed and does not affect the overall spirit of the reduction.

Let us fix any arbitrary order \succ_f among the candidates in $\mathcal{X} \uplus D$. For any subset $A \subset \mathcal{X} \uplus D$, let \vec{A} be the ordering among the candidates in A as defined in \succ_f and \overleftarrow{A} the reverse order of \vec{A} . For each $i \in [n]$, we add two votes v_i^j and $v_i^{j'}$ as follows for every $j \in [4]$. Let ℓ denote $|D| = 3n - 1$. Also, for $d \in D$, let $D_i, D_{\ell/2} \subset D \setminus \{d\}$ be such that:

$$|D_i| = \ell/2 + n + 1 - \lceil X_i/2 \rceil \text{ and } |D_{\ell/2}| = \ell/2.$$

$$v_i^j : \begin{cases} c \succ p \succ d \succ \overline{\mathcal{C} \setminus (\{d, c, p, x_i\} \uplus D_i)} \succ x_i \succ \overrightarrow{D_i} & \text{for } 1 \leq j \leq 2 \\ x_i \succ \overleftarrow{D_i} \succ \overline{\mathcal{C} \setminus (\{d, c, p, x_i\} \uplus D_i)} \succ c \succ p \succ d & \text{for } 3 \leq j \leq 4 \end{cases}$$

$$v_i^{j'} : \begin{cases} c \succ p \succ d \succ \overline{\mathcal{C} \setminus (\{d, c, p, x_i\} \uplus D_{\ell/2})} \succ x_i \succ \overrightarrow{D_{\ell/2}} & \text{for } 1 \leq j' \leq 2 \\ x_i \succ \overleftarrow{D_{\ell/2}} \succ \overline{\mathcal{C} \setminus (\{d, c, p, x_i\} \uplus D_{\ell/2})} \succ c \succ p \succ d & \text{for } 3 \leq j' \leq 4 \end{cases}$$

It is convenient to view the votes corresponding to $j = 3, 4$ as a near-reversal of the votes in $j = 1, 2$ (except for candidates c, d and x_i). Let $\mathcal{P}_1 = \{v_i^j, v_i^{j'} : i \in [n], j \in [4]\}$. Since there are $8n$ votes in all, and c always appears immediately before p , it follows that the score of c is exactly $8n$ more than the score of the candidate p in \mathcal{P}_1 .

We also observe that the score of the candidate x_i is exactly $2(\ell + n + 1) - X_i = 8n - X_i$ more than the score of the candidate p in \mathcal{P}_1 for every $i \in [n]$ such that X_i is an even integer. On the other hand, the score of the candidate x_i is exactly $2(\ell + n + 1) - X_i - 1 = 8n - X_i - 1$ more than the score of the candidate p in \mathcal{P}_1 for every $i \in [n]$ such that X_i is an odd integer. Note that for $i' \in [n] \setminus \{i\}$, p and x_i receive the same Borda score from the votes $v_{i'}^j$ and $v_{i'}^{j'}$ (where $j, j' \in [4]$).

We now add the following two votes μ_1 and μ_2 .

$$\mu_1 : p \succ c \succ \text{others}$$

$$\mu_2 : p \succ c \succ \text{others}$$

Let $\mathcal{P} = \mathcal{P}_1 \uplus \{\mu_1, \mu_2\}$, $\mathcal{X}^o = \{x_i : i \in [n], X_i \text{ is odd}\}$, and $\mathcal{X}^e = \mathcal{X} \setminus \mathcal{X}^o$. We recall that the distinguished candidate is p . The tie-breaking rule is according to the order $\mathcal{X}^o \succ p \succ \text{others}$. We claim that the FRUGAL-BRIBERY instance $(\mathcal{C}, \mathcal{P}, p)$ is equivalent to the PERMUTATION SUM instance (X_1, \dots, X_n) .

In the forward direction, suppose there exist two permutations π and σ of $[n]$ such that $\pi(i) + \sigma(i) = X_i$ for every $i \in [n]$. We replace the votes μ_1 and μ_2 with respectively μ'_1 and μ'_2 as follows.

$$\mu'_1 : p \succ D \succ x_{\pi^{-1}(n)} \succ x_{\pi^{-1}(n-1)} \succ \dots \succ x_{\pi^{-1}(1)} \succ c$$

$$\mu'_2 : p \succ D \succ x_{\sigma^{-1}(n)} \succ x_{\sigma^{-1}(n-1)} \succ \dots \succ x_{\sigma^{-1}(1)} \succ c$$

We observe that, the candidates c and every $x \in \mathcal{X}^e$ receive same score as p , every candidate $x' \in \mathcal{X}^o$ receives 1 score less than p , and every candidate in D receives less score than p in $\mathcal{P}_1 \uplus \{\mu'_1, \mu'_2\}$. Hence p wins in $\mathcal{P}_1 \uplus \{\mu'_1, \mu'_2\}$ due to the tie-breaking rule. Thus $(\mathcal{C}, \mathcal{P}, p)$ is a YES instance of FRUGAL-BRIBERY.

To prove the other direction, suppose the FRUGAL-BRIBERY instance is a YES instance. Notice that the only vulnerable votes are μ_1 and μ_2 . Let μ'_1 and μ'_2 be two votes such that the candidate p wins in the profile $\mathcal{P}_1 \uplus \{\mu'_1, \mu'_2\}$. We assume, without loss of generality, that candidate p is placed at the first position in both μ'_1 and μ'_2 . Since c receives $8n$ scores more than p in \mathcal{P}_1 , c must be placed at the last position in both μ'_1 and μ'_2 since otherwise p cannot win in $\mathcal{P}_1 \uplus \{\mu'_1, \mu'_2\}$. We also assume, without loss of generality, that every candidate

in D is preferred over every candidate in \mathcal{X} since otherwise, if $x \succ d$ in either μ'_1 or μ'_2 for some $x \in \mathcal{X}$ and $d \in D$, then we can exchange the positions of x and d and p continues to win since no candidate in D receives more score than p in \mathcal{P}_1 . Hence, every $x \in \mathcal{X}$ receives some score between 1 and n in both the μ'_1 and μ'_2 . Let us define two permutations π and σ of $[n]$ as follows. For every $i \in [n]$, we define $\pi(i)$ and $\sigma(i)$ to be the scores the candidate x_i receives in μ'_1 and μ'_2 respectively. The fact that π and σ , as defined above, is indeed a permutation of $[n]$ follows from the structure of the votes μ'_1, μ'_2 and the Borda score vector. Since p wins in $\mathcal{P}_1 \uplus \{\mu'_1, \mu'_2\}$, we have $\pi(i) + \sigma(i) \leq X_i$. We now have the following.

$$n(n+1) = \sum_{i=1}^n (\pi(i) + \sigma(i)) \leq \sum_{i=1}^n X_i = n(n+1)$$

Hence, we have $\pi(i) + \sigma(i) = X_i$ for every $i \in [n]$ and thus (X_1, \dots, X_n) is a YES instance of PERMUTATION SUM. \square

We will use Lemma 1 in subsequent proofs, which has been shown before (see, for instance, the work of Baumeister et al. [BRR11] and Dey et al. [DMN16b]).

Lemma 1. *Let $\mathcal{C} = \{c_1, \dots, c_m\} \uplus D$, ($|D| > 0$) be a set of candidates and $\vec{\alpha}$ a normalized score vector of length $|\mathcal{C}|$. Then, for any given $\mathbf{X} = (X_1, \dots, X_m) \in \mathbb{Z}^m$, there exists $\lambda \in \mathbb{R}$ and a voting profile such that the $\vec{\alpha}$ -score of c_i is $\lambda + X_i$ for all $1 \leq i \leq m$, and the score of candidates $d \in D$ is less than λ . Moreover, the number of votes is $O(\text{poly}(|\mathcal{C}| \cdot \sum_{i=1}^m |X_i|))$, where $|X_i|$ is the absolute value of X_i .*

Note that the number of votes used in Lemma 1 is polynomial in m if $|D|$ and $|X_i|$ are polynomials in m for every $i \in [m]$, which indeed is the case in all our proofs that use Lemma 1. Hence, the reductions in the proofs that use Lemma 1 run in polynomial time.

We now show the results for various classes of scoring rules. To this end, we reduce from the EXACT-COVER-BY-3-SETS (X3C) problem, which is known to be NP-complete [GJ79]. The X3C problem is defined as follows.

X3C

Input: A universe U and t subsets $S_1, \dots, S_t \subset U$ with $|S_i| = 3 \forall i \in [t]$.

Question: Does there exist an index set $I \subseteq [t]$ with $|I| = |U|/3$ such that $\uplus_{i \in I} S_i = U$?

We denote an arbitrary instance of X3C by $(U, \{S_1, \dots, S_t\})$.

Theorem 2. *The FRUGAL-\$BIBERY problem is NP-complete for the k -approval voting rule for any constant $k \geq 3$, even if the price of every vulnerable vote is either 1 or ∞ .*

Proof. The problem is clearly in NP. To show NP-hardness, we reduce an arbitrary instance of X3C to FRUGAL-\$BIBERY. Let $(U, \{S_1, \dots, S_t\})$ be an instance of X3C. We define a FRUGAL-\$BIBERY instance as follows. The candidate set is:

$$\mathcal{C} = \mathcal{U} \uplus \mathcal{D} \uplus \{p, q\}, \text{ where } |\mathcal{D}| = k - 1$$

For each $S_i, 1 \leq i \leq t$, we add a vote v_i as follows.

$$v_i : S_i \succ \mathcal{D} \succ p \succ q \succ \text{others}$$

By Lemma 1, we can add $\text{poly}(|\mathcal{U}|)$ many additional votes to ensure the following scores (denoted by $s(\cdot)$).

- $s(q) = s(p) + |\mathcal{U}|/3$
- $s(x) = s(p) + |\mathcal{U}|/3 + 1, \forall x \in \mathcal{U}$
- $s(d) < s(p) - |\mathcal{U}|, \forall d \in \mathcal{D}$

The tie-breaking rule is “ $p \succ \text{others}$ ”. The winner is q . The distinguished candidate is p and thus all the votes in $\{v_i : 1 \leq i \leq t\}$ are vulnerable. The price of every v_i is 1 and the price of every other vulnerable vote is ∞ . The budget is $|\mathcal{U}|/3$. This completes the construction. We now prove that the two instances are equivalent.

In the forward direction, let us suppose that there exists an index set $I \subseteq [t]$ with $|I| = |\mathcal{U}|/3$ such that $\uplus_{i \in I} S_i = \mathcal{U}$. We replace the votes v_i with $v'_i, i \in I$, which are defined as follows.

$$v'_i : \underbrace{p \succ \mathcal{D}}_{k \text{ candidates}} \succ \text{others}$$

This makes the score of p not less than the score of any other candidate and thus p wins.

To prove the result in the other direction, let us suppose that the FRUGAL-\$BIBERY instance is a YES instance. Then there exists $\mathcal{V} \subset \{v_i : 1 \leq i \leq t\}$ with $|\mathcal{V}| = |\mathcal{U}|/3$ such that no vote in $\{v_i : 1 \leq i \leq t\} \setminus \mathcal{V}$ has been changed by the briber. Let the vote that replaces $v \in \mathcal{V}$ be v' and let $\mathcal{V}' = \{v' : v \in \mathcal{V}\}$. We assume, without loss of generality, that the candidate p is placed within the first k positions of every vote $v' \in \mathcal{V}'$. Hence, the final score of the candidate p is $s(p) + |\mathcal{U}|/3$. We observe that, in every vote $v'_i \in \mathcal{V}'$, the candidate q and the corresponding S_i should not be placed within the top k positions since $s(p) + |\mathcal{U}|/3 = s(q)$ and $s(p) + |\mathcal{U}|/3 = s(x) - 1$ for every $x \in \mathcal{U}$. We claim that the S_i 's corresponding to the v_i 's in \mathcal{V} form an exact set cover. Indeed, otherwise, there will be a candidate $x \in \mathcal{U}$, whose score never decreases which contradicts the fact that p wins the election since $s(p) + |\mathcal{U}|/3 = s(x) - 1$. \square

We next present a similar result for the k -veto voting rule.

Theorem 3. *The FRUGAL-\$BIBERY problem is NP-complete for the k -veto voting rule for any constant $k \geq 3$, even if the price of every vulnerable vote is either 1 or ∞ .*

Proof. The problem is clearly in NP. To show NP-hardness, we reduce an arbitrary instance of X3C to FRUGAL-\$BRIBERY. Let $(U, \{S_1, S_2, \dots, S_t\})$ be any instance of X3C. We define a FRUGAL-\$BRIBERY instance as follows. The candidate set is:

$$\mathcal{C} = U \uplus Q \uplus \{p, a_1, a_2, a_3, d\}, \text{ where } |Q| = k - 3$$

For each $S_i, 1 \leq i \leq t$, we add a vote v_i as follows.

$$v_i : p \succ \text{others} \succ \underbrace{S_i \succ Q}_{k \text{ candidates}}$$

By Lemma 1, we can add $\text{poly}(|U|)$ many additional votes to ensure following scores (denoted by $s(\cdot)$).

- $s(p) > s(d), s(p) = s(x) + 2, \forall x \in U$
- $s(p) = s(q) + 1, \forall q \in Q$
- $s(p) = s(a_i) - |U|/3 + 1, \forall i = 1, 2, 3$

The tie-breaking rule is " $a_1 \succ \dots \succ p$ ". The winner is a_1 . The distinguished candidate is p and thus all the votes in $\{v_i : 1 \leq i \leq t\}$ are vulnerable. The price of every v_i is one and the price of any other vote is ∞ . The budget is $|U|/3$. We claim that the two instances are equivalent.

In the forward direction, suppose there exists an index set $I \subseteq \{1, \dots, t\}$ with $|I| = |U|/3$ such that $\bigcup_{i \in I} S_i = U$. We replace the votes v_i with $v'_i, i \in I$, which are defined as follows.

$$v'_i : \text{others} \succ \underbrace{a_1 \succ a_2 \succ a_3 \succ Q}_{k \text{ candidates}}$$

The score of each a_i decreases by $|U|/3$ and their final scores are $s(p) - 1$, since the score of p is not affected by this change. Also the final score of each $x \in U$ is $s(p) - 1$ since I forms an exact set cover. This makes p win the election.

To prove the result in the other direction, suppose the FRUGAL-\$BRIBERY instance is a YES instance. Then, notice that there will be exactly $|U|/3$ votes in $v_i, 1 \leq i \leq t$, where every $a_j, j = 1, 2, 3$, should come in the last k positions since $s(p) = s(a_j) - |U|/3 + 1$ and the budget is $|U|/3$. Notice that candidates in Q must not be placed within top $m - k$ positions since $s(p) = s(q) + 1$, for every $q \in Q$. Hence, in the votes that have been changed, a_1, a_2, a_3 and all the candidates in Q must occupy the last k positions. We claim that the S_i 's corresponding to the v_i 's that have been changed must form an exact set cover. If not, then, there must exist a candidate $x \in U$ and two votes v_i and v_j such that, both v_i and v_j have been replaced by $v'_i \neq v_i$ and $v'_j \neq v_j$ and the candidate x was present within the last k positions in both v_i and v_j . This makes the score of x at least the score of p which contradicts the fact that p wins. \square

We now show that there exists a polynomial time algorithm for the FRUGAL-\$BIBERY problem for the k-approval, simplified Bucklin, and plurality with runoff voting rules, when the budget is a constant. The result below follows from the existence of a polynomial time algorithm for the COALITIONAL-MANIPULATION problem for these voting rules for any number of manipulators [XZP⁺09].

Theorem 4. *The FRUGAL-\$BIBERY problem is in P for the k-approval, simplified Bucklin, and plurality with runoff voting rules, if the budget is a constant.*

Proof. Let the budget b be a constant. Then, at most b many vulnerable votes whose price is not zero can be changed since the prices are assumed to be in \mathbb{N} . Notice that we may assume, without loss of generality, that all the vulnerable votes whose price is zero will be changed. We iterate over all the $O(n^b)$ many possible vulnerable vote changes and we can solve each one in polynomial time since the COALITIONAL-MANIPULATION problem is in P for these voting rules [XZP⁺09]. \square

We show that the FRUGAL-\$BIBERY problem is NP-complete for a wide class of scoring rules as characterized in the following result.

Theorem 5. *For any positional scoring rule r with score vectors $\{\vec{s}_i : i \in \mathbb{N}\}$, if there exists a polynomial function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every $m \in \mathbb{N}$, $f(m) \geq 2m$ and in the score vector $(\alpha_1, \dots, \alpha_{f(m)})$, there exists a $m \leq \ell \leq f(m) - 5$ satisfying the following condition:*

$$\alpha_i - \alpha_{i+1} = \alpha_{i+1} - \alpha_{i+2} > 0, \forall \ell \leq i \leq \ell + 3$$

then the FRUGAL-\$BIBERY problem is NP-complete for r even if the price of every vulnerable vote is either 1 or ∞ .

Proof. The problem is clearly in NP. To show NP-hardness, we reduce an arbitrary instance of X3C to FRUGAL-\$BIBERY. Let $(U, \{S_1, \dots, S_t\})$ be an instance of X3C. We define a FRUGAL-\$BIBERY instance as follows. Let us consider the score vector $(\alpha_1, \dots, \alpha_{f(|U|)})$. Since the scoring rules remain unchanged if we multiply every α_i by any constant $\lambda > 0$ and/or add any constant μ , we can assume the following without loss of generality.

$$\alpha_i - \alpha_{i+1} = \alpha_{i+1} - \alpha_{i+2} = 1, \forall \ell \leq i \leq \ell + 3$$

The candidate set is:

$$\mathcal{C} = U \uplus Q \uplus \{p, a, d\}, \text{ where } |Q| = f(|U|) - |U| - 4 \text{ and } Q = \{q_1, q_2, \dots, q_{|Q|}\}$$

Let us fix any arbitrary order \succ_f among the candidates in $U \uplus Q$. For any subset $A \subset U \uplus Q$, let \vec{A} be the ordering among the candidates in A as defined in \succ_f . For each $S_i = \{x, y, z\}$, $1 \leq i \leq t$, we add a vote v_i as follows.

$$v_i : p \succ d \succ \overrightarrow{\text{others}} \succ \underbrace{a \succ x \succ y \succ z \succ q_1 \succ q_2 \succ \dots \succ q_{f(|U|)-\ell-4}}_{\ell \text{ candidates}}$$

By Lemma 1, we can add $\text{poly}(|U|)$ many additional votes to ensure the following scores (denoted by $s(\cdot)$) in the resulting profile (including the votes $v_i, i \in [t]$). Note that the proof of Lemma 1 by Baumeister et al. [BRR11] also works for the normalization of α defined in the beginning of the proof.

- $s(d) < s(p)$
- $s(x) = s(p) - 2, \forall x \in U$
- $s(a) = s(p) + |U|/3 - 1$
- $s(q) = s(p) - 1, \forall q \in Q$

The tie-breaking rule is “ $\dots \succ p$.” The candidate a wins. The distinguished candidate is p . The price of every v_i is 1 and the price of every other vulnerable vote is ∞ . The budget is $|U|/3$. We claim that the two instances are equivalent.

In the forward direction, there exists an index set $I \subseteq [t], |I| = |U|/3$, such that $\bigcup_{i \in I} S_i = U$. We replace the votes v_i with $v'_i, i \in I$, which are defined as follows.

$$v'_i : p \succ d \succ \overrightarrow{\text{others}} \succ x \succ y \succ z \succ a \succ q_1 \succ q_2 \succ \dots \succ q_{f(|U|)-\ell-4}$$

This makes the score of p at least one more than the score of every other candidate and thus p wins.

To prove the result in the other direction, let us suppose that the FRUGAL-\$BIBERY instance is a YES instance. Then there exists $\mathcal{V} \subset \{v_i : 1 \leq i \leq t\}$ with $|\mathcal{V}| = |U|/3$ such that no vote in $\{v_i : 1 \leq i \leq t\} \setminus \mathcal{V}$ has been changed by the briber. Let the vote that replaces $v \in \mathcal{V}$ be v' and let $\mathcal{V}' = \{v' : v \in \mathcal{V}\}$. Let the resulting profile be \mathcal{P}' . We first claim that the candidate $q_{f(|U|)-\ell-4}$ is placed at the last position of every $v' \in \mathcal{V}'$. Indeed, otherwise the score of the candidate $q_{\ell-4}$ is not less than the score of p in \mathcal{P}' which contradicts our assumption that the candidate p wins in \mathcal{P}' since the tie-breaking rule is “ $\dots \succ p$.” Given $q_{f(|U|)-\ell-4}$ is placed at the last position of every $v' \in \mathcal{V}'$, we observe, by the same argument applied for the candidate $q_{f(|U|)-\ell-5}$, that the candidate $q_{f(|U|)-\ell-5}$ is placed in the second last position of every $v' \in \mathcal{V}'$. Arguing similarly all the way to the candidate q_1 we observe that the last $(f(|U|) - \ell - 4)$ positions of every $v' \in \mathcal{V}'$ will be $q_1 \succ q_2 \succ \dots \succ q_{f(|U|)-\ell-4}$. Since $s(p) = s(a) - |U|/3 + 1$ and the tie-breaking rule is “ $\dots \succ p$,” the candidate a must be placed at the $(l + 4)^{\text{th}}$ position in every $v' \in \mathcal{V}'$. Hence, for every $i \in [t]$ such that $v'_i \in \mathcal{V}'$, if $S_i = \{x, y, z\}$, then the scores of the candidates x, y , and z increase by at least 1 each. We claim that $\bigcup_{i: v'_i \in \mathcal{V}'} S_i = U$. If not, then there must exist a candidate $x \in U$ whose score has increased by at least 2 contradicting the fact that p wins in \mathcal{P}' . \square

For the sake of concreteness, an example of a function f , stated in Theorem 5, that works for the Borda voting rule is $f(m) = 2m$. Theorem 5 shows that the FRUGAL-\$BIBERY problem is intractable for the Borda voting rule. However, the following theorem shows the intractability of the UNIFORM-FRUGAL-\$BIBERY problem for the Borda voting rule, even in a very restricted setting. Theorem 6 below is proved by a reduction from the COALITION

MANIPULATION problem for the Borda voting rule for two manipulators which is known to be NP-complete [BNW11, DKNW11].

Theorem 6. *The UNIFORM-FRUGAL-\$BIBERY problem is NP-complete for the Borda voting rule, even when every vulnerable vote has a price of 1 and the budget is 2.*

Proof. The problem is clearly in NP. To show NP-hardness, we reduce an arbitrary instance of the COALITIONAL-MANIPULATION problem for the Borda voting rule with two manipulators to an instance of the UNIFORM-FRUGAL-\$BIBERY problem for the Borda voting rule. Let $(C, \succ^t, 2, p)$ be an arbitrary instance of the COALITIONAL-MANIPULATION problem for the Borda voting rule and $|C| = m$. The corresponding FRUGAL-\$BIBERY instance is as follows. The candidate set is:

$$C' = C \uplus \{d, q\}$$

For each vote $v_i \in \succ^t$, we add a vote v'_i as follows.

$$v'_i : v_i \succ d \succ q$$

Let $\overrightarrow{C \setminus \{p\}}$ is an arbitrary but fixed order of the candidates in $C \setminus \{p\}$. Corresponding to the two manipulators', we add two more votes v_1 and v_2 as follows.

$$v_1, v_2 : \overrightarrow{C \setminus \{p\}} \succ d \succ p \succ q$$

Let $s(\cdot)$ and $s'(\cdot)$ be the score functions for the COALITIONAL-MANIPULATION and the UNIFORM-FRUGAL-\$BIBERY instances respectively. We assume that $s(x) < s(p) + 2(m-1)$ for every $x \in C \setminus \{p\}$ since otherwise the COALITIONAL-MANIPULATION instance is a trivial NO instance. We now add more votes to ensure following score differences in the resulting UNIFORM-FRUGAL-\$BIBERY instance.

$$s'(p) = \lambda + s(p) - 2, s'(x) = \lambda + s(x) \text{ for every } x \in C \setminus \{p\},$$

$$s'(q) = s'(p) + 2m - 1, s'(p) > s'(d) + 2m \text{ for some } \lambda \in \mathbb{Z}$$

This will be achieved as follows. For any two arbitrary candidates a and b , the following two votes increase the score of a by one more than the rest of the candidates except b whose score increases by one less. This construction has been used before [XCP10, DKNW11].

$$\begin{aligned} a \succ b \succ \overrightarrow{C \setminus \{a, b\}} \\ \overleftarrow{C \setminus \{a, b\}} \succ a \succ b \end{aligned}$$

Also, we can ensure that candidate p is always in $(^{m-1/2}, ^{m+1/2})$ positions and the candidate q never *immediately* follows p in these new votes. The tie-breaking rule is “ $q \succ \text{others} \succ p$.” The candidate q is the winner since $s'(q) = s'(p) + 2m - 1 \geq s'(x)$ for

every $x \in C' \setminus \{p, q\}$. The distinguished candidate is p . The price of every vulnerable vote is one and the budget is two. We claim that the two instances are equivalent.

In the forward direction, suppose the COALITIONAL-MANIPULATION instance is a YES instance. Let u_1, u_2 be the manipulators' votes that make p win. In the FRUGAL-\$BIBERY instance, we replace v_i by $v'_i : p \succ d \succ (u_i \setminus \{p\}) \succ q$ for $i = 1, 2$. This makes p win the election.

In the reverse direction, recall that in all the vulnerable votes except v_1 and v_2 , the candidate q never *immediately* follows candidate p . Therefore, changing any of these votes can never make p win the election since $s'(q) = s'(p) - 2m + 1$ and the budget is two. Hence, the only way p can win the election, if at all possible, is by changing the votes v_1 and v_2 . Let a vote v'_i replaces v_i for $i = 1, 2$. We can assume, without loss of generality, that p and d are at the first and the second positions respectively in both v'_1 and v'_2 . Let u_i be the order v'_i restricted only to the candidates in C . This makes p the unique winner of the COALITIONAL-MANIPULATION instance since $s'(p) = \lambda + s(p) - 2$, $s'(x) = \lambda + s(x)$ for every $x \in C$ and the tie-breaking rule is " $q \succ \text{others} \succ p$." \square

4. Results for Weighted Elections

Now we turn our attention to weighted elections. As before, we begin with some easy observations that follow from known results.

Observation 4. *The FRUGAL-BIBERY problem is in P for the maximin and the Copeland voting rules for three candidates.*

Proof. When we have 3 candidates, the COALITIONAL MANIPULATION problem is in P for the maximin and the Copeland voting rules [CSL07]. Hence, the result follows from Proposition 2. \square

Using the proof of Theorem 6 in Conitzer et al. [CSL07], we can obtain the following.

Observation 5. *Assume we have only 3 candidates. Then the FRUGAL-BIBERY problem is NP-complete for every scoring rule except plurality.*

Theorem 7. *The FRUGAL-BIBERY problem is in P for the plurality voting rule.*

Proof. Let p be the distinguished candidate of the campaigner. We greedily replace every vulnerable vote by the vote $p \succ \text{others}$. The correctness follows from the fact that plurality only accounts for candidates in the top position, and the strategy described is therefore the best possible for candidate p . \square

Our hardness results in this section are based on the PARTITION problem, which is known to be NP-complete [GJ79], and is defined as follows.

PARTITION

Input: A finite multi-set W of positive integers with $\sum_{w \in W} w = 2K$.

Question: Does there exist a subset $W' \subset W$ such that $\sum_{w \in W'} w = K$?

An arbitrary instance of PARTITION is denoted by $(W, 2K)$. We define another problem which we call $\frac{1}{4}$ -PARTITION as below. We prove that $\frac{1}{4}$ -PARTITION is also NP-complete. We will use this fact in the proof of Theorem 10.

$\frac{1}{4}$ -PARTITION

Input: A finite multi-set W of positive integers with $\sum_{w \in W} w = 4K$.

Question: Does there exist a subset $W' \subset W$ such that $\sum_{w \in W'} w = K$?

An arbitrary instance of $\frac{1}{4}$ -PARTITION is denoted by $(W, 4K)$.

Lemma 2. $\frac{1}{4}$ -PARTITION problem is NP-complete.

Proof. The problem is clearly in NP. To show NP-hardness, we reduce the PARTITION problem to it. Let $(W, 2K)$ be an arbitrary instance of the PARTITION problem. We can assume, without loss of generality, that $2K \notin W$, since otherwise the instance is trivially a *no* instance. The corresponding $\frac{1}{4}$ -PARTITION problem instance is defined by $(W_1, 4K)$, where $W_1 = W \cup \{2K\}$. We claim that the two instances are equivalent. Suppose the PARTITION instance is a YES instance and thus there exists a set $W' \subset W$ such that $\sum_{w \in W'} w = K$. This W' gives a solution to the $\frac{1}{4}$ -PARTITION instance. To prove the result in the other direction, suppose there is a set $W' \subset W_1$ such that $\sum_{w \in W'} w = K$. This W' gives a solution to the PARTITION problem instance since $2K \notin W'$. \square

Our hardness reductions in this section have the following overall approach. We first introduce vulnerable votes corresponding to the numbers in the instance of PARTITION, and the weights and prices of these votes are tightly correlated with the corresponding numbers in the PARTITION instance. We then introduce auxiliary votes, that are typically not vulnerable, but are crafted in such a way that the distinguished candidate lags behind the current winner — and the differential can only be compensated by changing *exactly* half the weight of the vulnerable votes. In the case of FRUGAL-\$BRIBERY, this is relatively easy to achieve: the score difference can be used to create the requirement that the total weight of the changed votes is at least K , while the budget can be used to enforce that the total cost of the changed votes is at most K , which leads us naturally to the desired partition. We see this in play in Theorem 8, for the plurality voting rule. For all the other rules, since we don't have costs, a more delicate argument is required to enforce the two-sided dynamic of the affected votes.

In the rest of this section, we present the hardness results in weighted elections for the following voting rules: plurality, maximin, STV, Copeland ^{α} , and simplified Bucklin. For plurality, recall that the FRUGAL-BRIBERY problem is in P, and we will show that FRUGAL-\$BRIBERY is NP-complete. For all the other rules, we will establish that even FRUGAL-BRIBERY is NP-complete.

Theorem 8. *The FRUGAL-\$BRIBERY problem is NP-complete for the plurality voting rule for three candidates.*

Proof. The problem is clearly in NP. We reduce an arbitrary instance of PARTITION to an instance of FRUGAL-\$BIBERY for the plurality voting rule. Let $(W, 2K)$, with $W = \{w_1, \dots, w_n\}$ and $\sum_{i=1}^n w_i = 2K$, be an arbitrary instance of the PARTITION problem. In the reduced instance, we introduce three candidates, namely, p , a , and b . The distinguished candidate is p . We will now add votes in such a way that makes b win the election.

For every $i \in [n]$, we have one vote $a \succ p \succ b$ of both weight and price w_i . We have two votes $b \succ p \succ a$ of weight $3K$ each (we do not need to define the price of this vote since it is non-vulnerable). We also have one vote $p \succ a \succ b$ of both weight and price $2K + 1$. This finishes the description of the votes. We observe that candidate b wins the plurality election with plurality score $3K$. The tie-breaking rule is “ $a \succ b \succ p$.” We define the budget to be K . We claim that the two instances are equivalent.

In the forward direction, suppose there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. We change the votes corresponding to the weights in W' to $p \succ a \succ b$. This makes p win the election with a plurality score of $3K + 1$.

To prove the other direction, for p to win, its score must increase by at least K . Also, the prices ensure that p 's score can increase by at most K . Hence, p 's score must increase by exactly by K and the only way to achieve this is to increase its score by changing the votes corresponding to the weights in W . Thus, p can win only if there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. \square

Next we show the hardness result for the maximin voting rule.

Theorem 9. *The FRUGAL-BIBERY problem is NP-complete for the maximin voting rule for 4 candidates.*

Proof. The problem is clearly in NP. We reduce an arbitrary instance of PARTITION to an instance of FRUGAL-BIBERY for the maximin voting rule. Let $(W, 2K)$, with $W = \{w_1, \dots, w_n\}$ and $\sum_{i=1}^n w_i = 2K$, be an arbitrary instance of the PARTITION problem. The candidates in our FRUGAL-BIBERY instance are p , a , b , and c . For every $i \in [n]$, we have one vote $p \succ a \succ b \succ c$ of weight w_i . There is one vote $c \succ a \succ b \succ p$, one vote $b \succ c \succ a \succ p$, and one vote $a \succ c \succ b \succ p$ each of weight K . The election is summarized in Table 2. This finishes the description of the votes. The weighted majority graph induced by these votes are shown in Figure 1. We observe that candidate a wins since it is the Condorcet winner of the election. The tie-breaking rule is “ $p \succ a \succ b \succ c$.” The distinguished candidate is p . Let T denote the set of votes corresponding to the weights in W and the rest of the votes S . Notice that only the votes in T are vulnerable. We claim that the two instances are equivalent.

Pairwise Outcomes	a	b	c	p
a	—	4K	3K	3K
b	K	—	3K	3K
c	2K	2K	—	3K
p	2K	2K	2K	—

Pairwise Outcomes	a	b	c	p
a	—	3K	2K	3K
b	2K	—	3K	3K
c	3K	2K	—	3K
p	2K	2K	2K	—

Table 2: Every cell shows the number of voters who prefer the row candidate over the column candidate. The left table shows the case of the reduced election in Theorem 9. The right table shows the case of the modified election in the forward direction of the proof of Theorem 9. The green cells show the witness of worst-case pairwise elections for every row candidate.

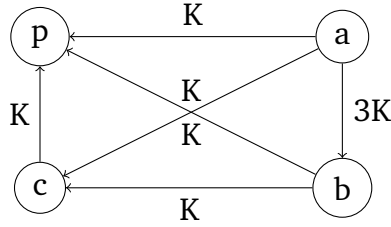


Figure 1: Weighted majority graph of the reduced instance in Theorem 9.

In the forward direction, suppose there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. We keep the votes corresponding to the weights in W' same as the original vote $p \succ a \succ b \succ c$. We change the rest of the votes in T to $p \succ b \succ c \succ a$. We observe from the weighted majority graph (shown in Figure 2) induced by these new set of votes that the maximin score of every candidate is $-K$ and thus due to the tie-breaking rule, p wins the election.

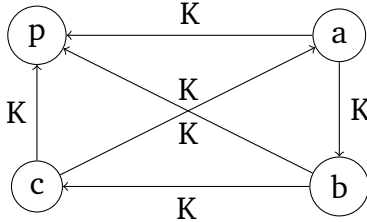


Figure 2: Weighted majority graph induced by the votes in the forward direction of the proof of Theorem 9.

To prove the result in the other direction, suppose there is a way to change the vulnerable votes, that is the votes in T , that makes p win the election. Let the new set of votes that replace T be T' . Without loss of generality, we can assume that all the votes in T' place p at the first position. Hence the maximin score of candidate p is $-K$. We first notice that the only way p could win is that the vertices a, b , and c must form a cycle in the weighted majority graph. Otherwise, one of a, b , and c will be a Condorcet winner and

thus the winner of the election. We also observe that the weight of every edge of the cycle consisting of a , b , and c only must be at least K . If not, then the maximin score of one of candidates in $\{a, b, c\}$ must be strictly more than $-K$. This contradicts our assumption that p wins since the maximin score of candidate p is fixed at $-K$.

Now, we claim that candidate b must defeat candidate c . Suppose not. Then, since the maximin score of p is fixed at $-K$, c must defeat b by a margin of at least K . Further, note that the margin must be exactly K since making c defeat b by a margin K requires c to be preferred over b in every vote in T' . On the other hand, a must defeat c by a margin of at least K (and thus exactly K), otherwise the maximin score of c will be more than $-K$. This implies that all the votes in T' must be $p \succ a \succ c \succ b$ which makes a defeat b . This is a contradiction since the vertices a , b , and c must form a cycle in the weighted majority graph. Hence b must defeat c by a margin of K .

Since b defeats c by a margin of K , every vote in T' is forced to prefer b over c . Without loss of generality, we assume that all the votes in T' are either $p \succ a \succ b \succ c$ or $p \succ b \succ c \succ a$, since whenever c is immediately after a , we can swap a and c and this will only reduce the score of a without affecting the score of any other candidates. If the total weight of the votes $p \succ a \succ b \succ c$ in T' is more than K , then $D_E(c, a) < K$, thereby making the maximin score of a more than the maximin score of p . If the total weight of the votes $p \succ a \succ b \succ c$ in T' is less than K , then $D_E(a, b) < K$, thereby making the maximin score of b more than the maximin score of p . Thus the total weight of the votes $p \succ a \succ b \succ c$ in T' should be exactly K which corresponds to a partition of W . \square

We now prove the hardness result for the STV voting rule.

Theorem 10. *The FRUGAL-BRIBERY problem is NP-complete for the STV voting rule for 3 candidates.*

Proof. The problem is clearly in NP. We reduce an arbitrary instance of $\frac{1}{4}$ -PARTITION to an instance of FRUGAL-BRIBERY for the STV voting rule. Let $(W, 4K)$, with $W = \{w_1, \dots, w_n\}$ and $\sum_{i=1}^n w_i = 4K$, be an arbitrary instance of the $\frac{1}{4}$ -PARTITION problem. The candidates in our FRUGAL-BRIBERY instance are p , a , and b . For every $i \in [n]$, we have a vote $p \succ a \succ b$ of weight w_i . We have one vote $a \succ p \succ b$ of weight $3K - 1$ and one vote $b \succ a \succ p$ of weight $2K$. This finishes the description of the votes. The tie-breaking rule is “ $a \succ b \succ p$.” We observe that the plurality score of candidates a , b , and p are $3K - 1$, $2K$, and $4K$ respectively in the resulting election. Hence candidate b gets eliminated in the first round. In the second round, the plurality score of candidates a and p are $5K - 1$ and $4K$ respectively. Hence candidate a wins the STV election. The distinguished candidate is p . Let T denote the set of votes corresponding to the weights in W and the rest of the votes be S . Notice that only the votes in T are vulnerable. We claim that the two instances are equivalent.

In the forward direction, suppose there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. We change the votes corresponding to the weights in W' to $b \succ p \succ a$. We do not change the rest of the votes in T . In the first round of the resulting profile, candidates a , b , and p receive a plurality score of $3K - 1$, $3K$, and $3K$ respectively. Hence candidate a gets

eliminated in the first round. In the second round, candidates b and p receive a plurality score of $3K$ and $6K - 1$ respectively and thus candidate p wins the election.

For the other direction, suppose there is a way to change the votes in T that makes p win the election. We first observe that candidate p can win only if p and b qualifies for the second round. Hence, the total weight of the votes in T that put b at the first position must be at least K . On the other hand, if the total weight of the votes in T that put b at the first position is strictly more than K , then p does not qualify for the second round and thus cannot win the election. Hence the total weight of the votes in T that put b at the first position must be exactly equal to K which constitutes a $\frac{1}{4}$ -partition of W . \square

For three candidates, the STV voting rule is the same as the plurality with runoff voting rule. Hence, we have the following corollary.

Corollary 1. *The FRUGAL-BRIBERY problem is NP-complete for the plurality with runoff voting rule for 3 candidates.*

We turn our attention to the Copeland $^\alpha$ voting rule next.

Theorem 11. *The FRUGAL-BRIBERY problem is NP-complete for the Copeland $^\alpha$ voting rule for 4 candidates for every $\alpha \in [0, 1)$.*

Proof. The problem is clearly in NP. We reduce an arbitrary instance of PARTITION to an instance of FRUGAL-BRIBERY for the Copeland $^\alpha$ voting rule. Let $(W, 2K)$, with $W = \{w_1, \dots, w_n\}$ and $\sum_{i=1}^n w_i = 2K$, be an arbitrary instance of the PARTITION problem. The candidates in our FRUGAL-BRIBERY instance are p, a, b, and c. For every $i \in [n]$, we have a vote $p \succ a \succ b \succ c$ of weight w_i . There are two votes $a \succ p \succ b \succ c$ and $c \succ b \succ a \succ p$ each of weight $K + 1$. This finishes the description of the votes. The tie-breaking rule is “ $a \succ b \succ c \succ p$.” The weighted majority graph induced by these votes are shown in Figure 3. We observe that candidate a wins since it is the Condorcet winner of the election. The distinguished candidate is p. Let T denote the set of votes corresponding to the weights in W and the rest of the votes be S . Notice that only the votes in T are vulnerable. We claim that the two instances are equivalent.

Pairwise Outcomes	a	b	c	p	Pairwise Outcomes	a	b	c	p
a	—	3K + 1	3K + 1	2K + 2	a	—	K + 1	K + 1	2K + 2
b	K + 1	—	3K + 1	K + 1	b	3K + 1	—	2K + 1	K + 1
c	K + 1	K + 1	—	K + 1	c	3K + 1	2K + 1	—	K + 1
p	2K	3K + 1	3K + 1	—	p	2K	3K + 1	3K + 1	—

Table 3: Every cell shows the number of voters who prefer the row candidate over the column candidate. The left table shows the case of the reduced election in Theorem 11. The right table shows the case of the modified election in the forward direction of the proof of Theorem 11. Green cells indicate that the row candidate defeats the column candidate in pairwise election. Yellow cells indicate that the row and the column candidate are tied in pairwise election.

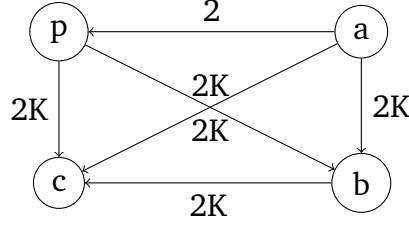


Figure 3: Weighted majority graph of the reduced instance in Theorem 11.

In the forward direction, suppose there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. We change the votes corresponding to the weights in W' to $p \succ c \succ b \succ a$. We change the rest of the votes in T to $p \succ b \succ c \succ a$. We observe from the weighted majority graph (shown in Figure 4) induced by these new set of votes that the Copeland ^{α} score of candidate p is 2 and the Copeland ^{α} score of every other candidate is strictly less than 2. Hence, candidate p wins the election.

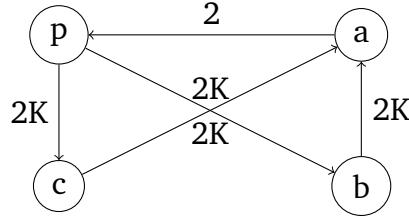


Figure 4: Weighted majority graph induced by the votes in the forward direction of the proof of Theorem 11.

For the other direction, suppose there is a way to change the votes in T that makes p win the election. Let the new set of votes that replace T be T' . Without loss of generality, we can assume that all the votes in T' place p at the top position. We claim that one of the three pairwise elections among a , b , and c must be a tie. Suppose not, then a must lose to both b and c , otherwise a wins the election due to the tie-breaking rule. Now consider the pairwise election between b and c . If b defeats c , then b wins the election due to the tie-breaking rule. If c defeats b , then c wins the election again due to the tie-breaking rule. Hence, one of the pairwise elections among a , b , and c must be a tie. Without loss of generality suppose a and b ties. However, then the total weight of the votes that prefer a to b in T' must be K which constitutes a partition of W . \square

Finally, we show that the FRUGAL-BRIBERY problem for the simplified Bucklin voting rule is NP-complete.

Theorem 12. *The FRUGAL-BRIBERY problem is NP-complete for the simplified Bucklin voting rule for 4 candidates.*

Proof. The problem is clearly in NP. We reduce an arbitrary instance of PARTITION to an instance of FRUGAL-BRIBERY for the simplified Bucklin voting rule. Let $(W, 2K)$, with

$W = \{w_1, \dots, w_n\}$ and $\sum_{i=1}^n w_i = 2K$, be an arbitrary instance of the PARTITION problem. The candidates in our FRUGAL-BRIBERY instance are p, a, b , and c . For every $i \in [n]$, we have one vote $p \succ a \succ b \succ c$ of weight w_i . There are two votes $a \succ b \succ p \succ c$ and $c \succ b \succ a \succ p$ each of weight K . This finishes the description of the votes. The tie-breaking rule is “ $p \succ a \succ b \succ c$.” Observe that:

- The candidates a and b get majority within the first two positions .
- The candidate p does *not* get majority within the first two positions.
- No candidate gets majority within the first position.

Therefore, candidate a wins due to the tie-breaking rule. We set the distinguished candidate as p . Let T denote the set of votes corresponding to the weights in W and the rest of the votes be S . Notice that only the votes in T are vulnerable. We claim that the two instances are equivalent.

In the forward direction, suppose there exists a $W' \subset W$ such that $\sum_{w \in W'} w = K$. We change the votes corresponding to the weights in W' to $p \succ c \succ b \succ a$. We keep the votes corresponding to the weights in $W \setminus W'$ same as the original ones. Now no candidate gets majority within first two positions and candidate p gets majority within first two positions. This makes p win the election with a simplified Bucklin score of 3 due to the tie-breaking rule.

To prove the result in the other direction, suppose there is a way to change the votes in T that makes p win the election. Let the new set of votes that replace T be T' . Without loss of generality, we can assume that all the votes in T' place p at the first position. We first notice that the simplified Bucklin score of p is already fixed at three. In the votes in T' , candidate b can never be placed at the second position since that will make the simplified Bucklin score of b to be two. Also the total weight of the votes in T' that place a in their second position can be at most K . The same holds for c . Hence, the total weight of the votes that place a in their second position will be exactly equal to K which constitutes a partition of W . \square

From Proposition 1, Observation 5, Theorem 8 to 12, and Corollary 1, we get the following.

Corollary 2. *When we have 3 candidates, the UNIFORM-FRUGAL-\$BRIBERY and the NONUNIFORM-FRUGAL-\$BRIBERY problems are NP-complete for the scoring rules except plurality, STV, and the plurality with runoff voting rules. When we have 4 candidates, the UNIFORM-FRUGAL-\$BRIBERY and the NONUNIFORM-FRUGAL-\$BRIBERY problems are NP-complete for the maximin, Copeland, and simplified Bucklin voting rules.*

5. Conclusion and Future Work

We have proposed and studied two important special cases of the \$BRIBERY problem where the briber is frugal. We have shown that the FRUGAL-BRIBERY problem can sometimes be polynomial time solvable even if the \$BRIBERY and the SWAP-BRIBERY problems

are NP-complete as observed for the k-approval and the k-veto voting rules for unweighted elections. This establishes success in finding important practical special cases of the sophisticated \$BRIBERY and SWAP-BRIBERY problems. We also proved that the FRUGAL-BRIBERY problem is NP-complete for the Borda voting rule and the FRUGAL-\$BRIBERY problem is NP-complete for all the voting rules studied here except the plurality and the veto voting rules for unweighted elections. The intractability results of the FRUGAL-\$BRIBERY problem and the FRUGAL-\$BRIBERY problem thereby subsumes and strengthens the hardness results for the \$BRIBERY problem. For the weighted election, we have shown that the simplest FRUGAL-BRIBERY problem also is NP-complete for all the voting rule studied in this paper except for the plurality voting rule even when the number of candidates is as small as 3 or 4. We find these results in the weighted elections both surprising and interesting.

An immediate future work is to resolve the open cases in Table 1. Another important direction for future work is to study these problems under various other settings. Notably, one might consider enhancing our proposed model further to account for constraints that arise in practical scenarios. For instance, we might want to restrict the campaigner's knowledge about the votes and/or the candidates who will actually turn up. The uncertainty can also arise from the voting rule that will eventually be used among a set of voting rules. Also, studying these problems when the pricing model for vulnerable votes is similar to swap bribery would be another interesting future direction. We believe that a game theoretic perspective of the problem may also yield valuable insights.

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